Cloud Publications
International Journal of Advanced Mathematics and Statistics
2017, Volume 3, Issue 1, pp. 68-72
ISSN 2348-5175, Crossref: 10.23953/cloud.ijams.329

Research Article

On ω-Convergence of *p*-Stacks

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Publication Date: 14 December 2017

DOI: https://doi.org/10.23953/cloud.ijams.329

Abstract We introduce the notion of ω-convergence of *p*-stacks and by using that notion we characterize the ω-interior, ω-closure, separation axioms and ω-irresoluteness on a topological space.
Keywords *Topological spaces; ω-open sets; ω-closed spaces*

1. Introduction

Generalized open sets play a very important role in General Topology and they are now the research topics of many topologists worldwide. Indeed a significant theme in General Topology and Real analysis concerns the various modified forms of continuity, separation axioms etc. by utilizing generalized open sets. Sundaram and Sheik John [5] introduced a new class of generalized open sets called ω-open sets into the field of topology. In this paper, we have introduced and study the notion of ω-convergence of p-stacks and by using that notion we characterize the ω-interior, ω-closure, separation axioms and ω-irresoluteness on a topological space. Also we have introduced a new notion of p-ω-compactness and investigate its properties in terms of ω-convergence of p-stacks.

2. Preliminaries

Throughout this paper, spaces always means topological spaces on which no separation axioms are assumed unless otherwise mentioned and $f : (X,\tau) \to (Y,\sigma)$ (or simply $f : X \to Y$) denotes a function f of a space (X,τ) into a space (Y,σ). Let A be a subset of a space X. The closure and the interior of A are denoted by Cl(A) and Int(A), respectively. A subset A of a space (X,τ) is called semi open [1] if $A \subset Cl(Int(A))$. A subset A of a space X is called ω-closed [5] if $Cl(A) \subset U$ whenever $A \subset U$ and U is semi-open in X. The complement of an ω-closed set is called an ω-open set. The family of all ω-open subsets of (X,τ) is denoted by $\omega(\tau)$. We set $\omega(X,x) = \{V \in \omega(\tau) | x \in V\}$ for $x \in X$. The union (resp. intersection) of all ω-open (resp. ω-closed) sets, each contained in (resp. containing) a set A in a space X is called the ω-interior (resp. ω-closure) of A and is denoted by ω Int(A) (resp. ω Cl(A)) [4]. A subset M(x) of a topological space X is called a ω-neighbourhood of a point $x \in X$ if there exists a ω-open set S such that $x \in S \subset M(x)$. Given a set X, a collection C of subsets of X is called a stack if $A \in C$ whenever $B \in C$ and $B \subset A$. A stack H on a set X is called a p-stack if it satisfies the following condition: (P) $A,B \in H \Rightarrow A \cap B \neq \emptyset$. Condition (P) is called the pairwise intersection property (P.I.P). A collection B of subsets of X with the P.I.P is called a p-stack base. For any collection B, we denote by $\langle B \rangle = \{A \subset X$: there exists $B \in B$ such that $B \subset A\}$ the stack generated by B, and if {B} is a p-stack base, then $\langle\{B\}\rangle$ is a p-stack. We will denote simply $\langle\{B\}\rangle = \langle B \rangle$. In case $x \in X$ and $B = \{x\}$, $\langle x \rangle$ is usually denoted by ˙x. Let pS(X) denote the collection of all p-stacks on X, partially ordered by inclusion. The maximal elements in pS(X) are called ultrapstacks is contained in an ultrapstack. For a function f : X → Y and H ∈ pS(X), the image stack f(H) in pS(Y) has p-stack base {f(H) : H ∈ H}. Likewise, if G ∈ pS(Y), f−1(G) denotes the p-stack on X generated by {f−1(G) : G ∈ G}.

Definition 2.1. Let (X,τ) be a topological space. A class {Gi} of ω-open subsets of X is said to be ω-open cover of X if each point in X belongs to atleast one Gi that is∪ i Gi = X.

Definition 2.2. A subset K of a nonempty set X is said to be ω-compact [4] relative to (X,τ) if every cover of K by ω-open sets of X has a finite subcover. We say that (X,τ) is ω-compact if X is ω-compact.

Definition 2.3. A topological space (X,τ) is said to be:

(i) ω-T1 [3] if for each pair of distinct points x and y of X, there exist ω-open sets U and V containing x and y, respectively such that y / ∈ U and x / ∈ V .
(ii) ω-T2 [3] if for each pair of distinct points x and y of X, there exist ω-open sets U and V such that x ∈ U, y ∈ V and U ∩V = ∅.
(iii) ω-regular [4] if for any closed set F ⊂ X and any point x ∈ X\F, there exist disjoint ω-open sets U and V such that x ∈ U and F ⊂ V .

Lemma 2.4. [2] For H ∈ pS(X), the following are equivalent:

(i) H is an ultrapstack.
(ii) If A∩H /= ∅ for all H ∈ H, then A ∈ H;
(iii) B ∈ H implies X\B ∈ H.

Theorem 2.5. [2] Let f : (X,τ) → (Y,σ) be a function and H ∈ pS(X).

(i) If H is a filter, so is f(H);
(ii) If H is an ultrafilter, so is f(H);
(iii) If H is an ultrapstack, so is f(H).

3. ω-convergence of p-stacks

Definition 3.1. Let X be a topological space, x ∈ X and let B(x) = {V ⊂ X: V is a ω-neighbourhood of x}. Then we call the family B(x) the ω-neighbourhood stack at x.

Definition 3.2. Let X be a topological space, x ∈ X and let B(x) = {V ⊂ X: V is a ω-neighbourhood of x}. Then we call the family B(x) the ω-neighbourhood stack at x.

Theorem 3.3. Let (X,τ) be a topological space. Then we have the following

(i) x ω-converges to x for all x ∈ X.
(ii) If F ω-converges to x and F ⊂ G for F,G ∈ pS(X), then G ω-converges to x.
(iii) If both F and G are p-stacks ω-converging to x, then F ∩ G ω-converges to x.
(iv) If p-stacks Fi ω-converge to x for all i ∈ J, then ∩Fi ω-converges to x.

Proof. Follows from the definitions.

Theorem 3.4. Let (X,τ) be a topological space and $A \subset X$. Then the following are equivalent:

(i) $x \in \omega\, Cl(A)$;
(ii) There is $F \in pS(X)$ such that $A \in F$ and F ω-converges to x;
(iii) For all $V \in B(x)$, $A \cap V \neq \emptyset$.

Proof. (i) $\Rightarrow$ (ii): Let x be an element in $bCl(A)$, then $U(x) \cap A = \emptyset$ for each ω-open $U(x)$ of x. Let $F = B(x) \cup \langle A \rangle$. Then the p-stack F ω-converges to x and $A \in F$. (ii) $\Rightarrow$ (iii): Let F be a p-stack and $A \in F$ and p-stack F ω-converge to x. Then $B(x) \subset F$. Thus since $B(x)$ is a p-stack, we get $U \cap A \neq \emptyset$ for all $U \in B(x)$. (iii) $\Rightarrow$ (i): It is obvious.

Theorem 3.5. Let (X,τ) be a topological space and $A \subset X$. Then the following are equivalent:

(i) $x \in \omega\, Int(A)$;
(ii) For every p-stack F ω-converging to x, $A \in F$;
(iii) $A \in B(x)$.

Proof. (i) $\Rightarrow$ (ii): Let x be an element in $bInt(A)$ and let F be a p-stack ω-converging to x. Since $x \in bInt(A)$, there is a ω-open subset U such that $x \in U \subset A$, so $A \in B(x)$. Thus by the definition of ω-convergence of p-stack, we can say $A \in F$. (ii) $\Rightarrow$ (iii): The ω-neighborhood stack $B(x)$ is always ω-converges to x. Thus by (ii), $A \in B(x)$. (iii) $\Rightarrow$ (i): It is obvious.

Now by using ω-convergence of p-stacks, we characterize the properties of ω-T1, ω-T2 and ω-regular induced by ω-open subsets on a topological space.

Theorem 3.6. Let (X,τ) be a topological space. Then the following statements are equivalent:

(i) (X,τ) is ω-T1;
(ii) $\cap B(x) = \{x\}$ for $x \in X$;
(iii) If $\dot{x}$ ω-converges to y, then $x = y$.

Proof. (i) $\Rightarrow$ (ii): Let y be an element in $\cap B(x)$, then $y \in U$ for each ω-open neighborhood U of x. Since X is ω-T1, we get $y = x$. (ii) $\Rightarrow$ (iii): Let $\dot{x}$ ω-converge to y. Since $B(y) \subset \dot{x}$, x is an element in $\cap B(y)$. Thus $x = y$. (iii) $\Rightarrow$ (i): Suppose that X is not ω-T1, then there are distinctx and y such that every ω-open neighborhood of x contains y. Thus $B(x) \subset \dot{y}$ and $\dot{y}$ ω-converges to x. This contradicts the hypothesis.

Theorem 3.7. Let (X,τ) be a topological space. Then the following statements are equivalent:

(i) (X,τ) is ω-T2;
(ii) Every ω-convergent p-stack F on X ω-converges to exactly one point;
(iii) Every ω-convergent ultrapstack F on X ω-converges to exactly one point.

Proof. (i) $\Rightarrow$ (ii): Suppose that X is ω-T2 and a p-stack F ω-converges to x. For any $y \neq x$, there are disjoint ω-open sets $U(x)$ and $U(y)$ containing x and y, respectively. Since $B(x) \subset F$ and F is a p-stack, both $U(x)$ and $X \backslash U(y)$ are elements of F. Thus F is not finer than (y), so F doesn't ω-converge to y. (ii) $\Rightarrow$ (iii): It is obvious. (iii) $\Rightarrow$ (i): Suppose that X is not ω-T2. Then there must exist x, y such that $U(x) \cap U(y) \neq \emptyset$ for every ω-open sets $U(x)$ and $U(y)$ of x and y, respectively. Let F be a ultrapstak finer than a p-stack $B(x) \subset B(y)$. Then F is finer than $B(x)$ and (y), so the ultrapstack F ω-converges to both x and y. This contradicts (ii). If (X,τ) is a topological space and $F \in pS(X)$, then $B = \{\omega\, Cl(F) : F \in F\}$ is a p-stack base on X, and the ω-closure p-stack generated by B is denoted by $\omega\, Cl(F)$.

Theorem 3.8. Let (X,τ) be a topological space. Then the following statements are equivalent:

(i) (X,τ) is ω-regular;
(ii) For every x ∈ X, B(x) = ω Cl(B(x));
(iii) If a p-stack F ω-converges to x, then the ω-closure p-stack ω Cl(F) ω-converges to x.

Proof. (i) ⇒ (ii): Let F be an element in B(x). There exists a ω-open neighborhood U(x) such that U(x) ⊂ F. Since X is ω-regular, there is a ω-open neighborhood W(x) of x such that W(x) ⊂ ω Cl(W(x)) ⊂ U(x) ⊂ F. Since ω Cl(W(x)) ∈ ω Cl(B(x)) and ω Cl(B(x)) is a pstack, F ∈ ω Cl(B(x)). (ii) ⇒ (iii): Let a p-stack F ω-converge to x. Then B(x) ⊂ F, and so ω Cl(B(x)) ⊂ ω Cl(F). By (ii), we get that ω Cl(F) ω-converges to x. (iii) ⇒ (i): Let U be a ω-open set containing x ∈ X. Since B(x) ω-converges to x, by (iii) ω Cl(B(x)) ω-converges to x, and so U ∈ ω Cl(B(x)). Then by the definition of the ω-closure of p-stacks, we can get a ω-open neighborhood V of x such that V ⊂ ω Cl(V) ⊂ U.

Definition 3.9. A function f: (X,τ) → (Y,σ) is said to be ω-irresolute [4] if f−1(V) is ω-closed (resp. ω-open) in X for every ω-closed (resp. ω-open) subset V of Y.

Theorem 3.10. Let X and Y be topological spaces. Then a function f : (X,τ) → (Y,σ) is ω-irresolute if and only if for each x in X and each ω-neighborhood U of f(x), there is a ω-neighborhood V of x such that f(V) ⊂ U. Now we get another characterization of the ω-irresolute function on a topological space using the notion of p-stacks.

Theorem 3.11. For a function f : (X,τ) → (Y,σ), the following statements are equivalent: (i) f is ω-irresolute; (ii) B(f(x)) ⊂ f(B(x)) for all x ∈ X; (iii) If a p-stack F ω-converges to x, then the image p-stack f(F) ω-converges to f(x).

Proof. (i) ⇒ (ii): Let V be any member of B(f(x)) in Y. Then there is a ω-open set W such that W ⊂ V. Since f is ω-irresolute, there exists a ω-open neighborhood U ∈ B(x) such that f(U) ⊂ W ⊂ V , thus V ∈ f(B(x)). (ii) ⇒ (iii): It is obvious. (iii) ⇒ (i): If f is not ω-irresolute, then for some x ∈ X, there is a ω-open neighborhood V ∈ B(f(x)) such that for all ω-open neighborhood U ∈ B(x), f(U) is not included in V. For all U ∈ B(x), since f(U)∩(Y\V)⊭ ∅, we get a p-stack F = f(B(x))∪<Y\V >. And since U ∩f−1(Y\V)⊭ ∅, also we get a p-stack G = B(x)∪f−1<Y\V > which ω-converges to x. But since f(G) is a finer p-stack than F and Y\V ∈ F, f(G) can't ω-converge to f(x), contradicting to (iii).

Now we introduce a new notion of p-ω-compactness by p-stacks and investigate the related properties.

Definition 3.12. Let (X,τ) be a topological space and A be a subset of X. A subset A of a topological space (X,τ) is p-ω-compact if every ultrapstack containing A ω-converges to a point in A. A topological space (X,τ) is p-ω-compact if X is p-ω-compact.

Let X = {a,b,c}. In case τ is the discrete topology, let H be an ultrapstack containing a p-stack F generated by {{a,b},{b,c},{a,c}}. Then it doesn't ω-converge to any point in X. Thus the topological space (X,τ) is not p-ω-compact. But in case τ = {∅,{a},{b,c},X}, the topological space (X,τ) is p-ω-compact.

Theorem 3.13. If a topological space (X,τ) is p-ω-compact and A ⊂ X is ω-closed, then A is p-ω-compact.

Proof. Let F be an ultrapstack containing A. From Definition 3.12, there is x ∈ X such that F ω-converges to x. Thus B(x) ⊂ F, and since A ∈ F and F is a p-stack, A∩V⊭ ∅ for all V ∈ B(x). So by

Theorem 3.4, we can say x ∈ ω Cl(A) = A.

Theorem 3.14. The ω-irresolute image of a p-ω-compact set is p-ω-compact.

Proof. Let a function f : (X,τ) → (Y,σ) be ω-irresolute, let AX be p-ω-compact, and let H be an ultrapstack containing f(A). If G is an ultrapstack containing the p-stack base {f−1(H) : H ∈ H}∪<A>, then for some x ∈ A, G ω-converges to x, and H = f(G) ω-converges to f(x). Thus, f(A) is p-ω-compact.

Theorem 3.15. A topological space (X,τ) is p-ω-compact if and only if each ω-open cover of X has a two-element subcover.

Proof. Suppose H is an ultrapstack in X such that it doesn't ω-converge to any point in X. Then for each x ∈ X, there is a ω-open subset Ux ∈ B(x) such that Ux / ∈ H. By Lemma 2.4(iii), X\Ux ∈ H, for allx ∈ X. Thus U = {Ux : x ∈ X} is a ω-open cover of X. But U has no two-element subcover of X, for if U,V ∈ U and X ⊂ U ∪V , then (X\U)∩(X\V) = X\(U ∪V) = ∅, contradicting the assumption that H is a p-stack. Conversely, let U be a ω-open cover of X with no two-element subcover of X. Then B = {X\U: U ∈ U} is p-stack base, and any ultrapstack containing B cannot ω-converge to any point in X.

References

[1] Levine, N. 1963. Semi-open sets and semi-continuity in topological spaces. *Amer. Math. Monthly*, 70, pp.36-41.

[2] Kent, D.C. and Min, W.K. 2002. Neighbourhood spaces. *Internat. J. Math. Math. Sci.*, 32(7), pp.387-399.

[3] Maki, H., Sundaram, P. and Rajesh, N. Characterization of ω-T_0, ω-T_1 and ω-T_2 topological spaces (Under Preparation).

[4] Sheik John, M. 2002. *A study on generalizations of Closed sets and Continuous maps in topological and bitopological spaces*, Ph.D. Thesis, Bharathiyar University, Coimbatore, India.

[5] Sundaram, P. and Sheik John, M. 1995. *Weakly Closed sets and Weak Continuous maps in topological spaces*, Indian Science Congress, Calcutta, p.49.