Cloud Publications

Research Article　　　　　　　　　　　　　　　　　　**Open Access**

Construction of Super-Saturated Designs

Ameen Saheb Sk.[1] and Bhatra Charyulu N.Ch.[2]

[1]School of Mathematics and Statistics, University of Hyderabad, Hyderabad
[2]Department of Statistics, University College of Science, Osmania University, Hyderabad

OPEN ACCESS

Abstract A design which is having more number of factors than the number of design points is called a Super-saturated Design. In this paper, an attempt is made to propose two new series of constructions of super-saturated designs using mutual orthogonal Latin squares and balanced incomplete block designs and the methods are illustrated with suitable examples.
Keywords *Balanced Incomplete Block Design; Mutual Orthogonal Latin Squares; Super-Saturated Design*

1. Introduction

If the number of factors or factor combinations is more, and only few measured are interest to study, then eliminate the insignificant factor combinations which are not affecting much the response that is the loss of information due to the elimination of factors should be as minimum as possible. High-dimensional datasets makes many mathematical challenges in fitting and give rise to new theoretical developments, which reduces the time, cost, effort and complexity by reducing its dimension. To minimize the number of design points, identify the active factors for efficient utilization of resources. The knowledge of each and every main effect may not useful since insignificant factors are not usually of interest. Reduce the number of design points or chose the design with minimum design points and optimum. These designs reduce the experimental cost and time significantly due to their run size.

Scatterthwaite (1959) initially constructed a new class of balanced designs randomly with a property that number of design points is equal to the number of factors and Booth and Cox (1962) proposed a systematic method of construction of designs in which the number of factors exceeds the number of design points. Let 'p' be the number of factors and 'n' be the number of design points in a design. Then the saturated and super saturated designs can be defined as

Definition 1.1: A design X is said to be 'saturated' if the number of design points 'n' is equal to the number of factors 'p' plus one i.e. n=p+1.
Definition 1.2: A design X is said to be a super-saturated design, if the number of factors "p" is more than the number of design points 'n' i.e. p > n.

Designs satisfying the orthogonality are more preferred due to their optimality. If it is not possible to conduct the experiment with orthogonal design, search for designs that are near orthogonal. For such designs, the lack of orthogonality can be measured based on the covariance's.

Definition 1.3: A design X is said to be $E(s^2)$-optimal super-saturated if a super-saturated design possessing the property that the mean of s_{ij}^2 of all pairs (i, j) for (i≠ j) is minimum.

2. Literature Review on Super-Saturated Designs

After Booth and Cox (1962), several authors like Lin (1993), Nguyen (1996), Cheng (1997), Li and Wu (1997) Tang and Wu (1997), Deng, Lin and Wang (1999), Fang, Lin and Ma (2000), Liu and Zhang (2000), Lu and Sun (2001), Butler, Mead, Eskridge and Gilmour (2001), and Yamada and Lin (2002), Li and Lin (2003), Liu and Dean (2004), Fang, Gennian and Liu (2004), Aggarwal and Gupta (2004), Xu and Wu (2005), Koukouvinov, Mantos and Mylona (2007), Jones, Lin and Nachtsheim (2008), Nguyen and Cheng (2008), Sun, Lin and Liu (2011), Liu and Liu (2012), Gupta and Morales (2012), Chatterjee, K., Koukouvinos, C, Mantas, P and Skountzou, A (2012), Hung, C., Lin, D.K.J., and Liu M.Q. (2012), Mbegbu, J.I., and Todo, C.O. (2012), Liu, Y. and Liu, M.Q. (2013), Miller, A., and Tang, B, (2013), Ameen Saheb and Bhatracharyulu (2013, 2014) etc made attempts on the construction of super-saturated designs with their $E(s^2)$ optimality.

Lin (1993), Li and Wu (1997) motivated towards the construction of super-saturated designs through column wise, pair wise exchanges. They differ from the k-exchange algorithms in two aspects, one is, they exchange columns instead of rows of the design matrix and another one is, they employ a pair wise adjustment in the search for a better column. Deng, Lin and Wang (1999) studied the properties of super-saturated designs and proposed a criterion based on the projection property called resolution rank.

Fang, Lin, and Ma (2000) proposed a construction procedure by embedding a saturated orthogonal design into a uniform design of the same row size. They adopt the collapsing method from Addelman. The basic idea of the construction method is, to collapse a multi-level factor into several low-level factors, where they collapse U-type uniform designs. They proposed five criteria for comparing multi-level super-saturated designs.

Liu and Zhang (2000) proposed a general algorithm for the construction of $E(s^2)$ optimal super-saturated designs from cyclic BIBD. The general formula for the lower bound of a super-saturated design with 'm' factors with 'n' design points is $[n^2(m-n+1) / [(n-1)(m-1)]$. Lu and Sun (2001) proposed two criteria denoted by $E(s^2)$ and $\max\{(s^2)\}$ in the construction of multi-level super-saturated designs. Eskridge, Gilmour, Mead, Butler and Travnicek (2001) and Liu and Dean (2002) also considered cyclic generation of $E(s^2)$-optimal and nearly optimal super-saturated designs.

Fang, Gennian and Liu (2002) proposed a discrete discrepancy as a measure of uniformity for super-saturated designs and a lower bound of this discrepancy is obtained as a benchmark of design uniformity and also proposed construction procedure for uniform super-saturated by using resolvable BIBD along with their properties. Yamada and Lin (2002) suggested a construction method for mixed-level super-saturated designs consisting of two-level and three-level columns. The chi-square statistics is used for a measure of dependency of the design columns. The dependency properties for the newly constructed designs are derived and discussed. Li and Lin (2002) proposed variable selection procedure to screen active effects in the super-saturated designs via non-convex penalized least squares approach and empirical comparison with Bayesian variable selection approaches is made.

Liu and Dean (2004) proposed a class of super-saturated designs called k- Circulant super-saturated designs which can be obtained from cyclic development of a generator. This method is a generalization of Plackett-Burman, who introduced the use of cyclic generators for constructing orthogonal saturated designs.

Fang Kaitai, Gennian G.E., and Liu Minqian (2004) proposed a combinatorial approach called the packing method. They studied the connection between orthogonal arrays and resolvable packing designs for constructing optimal super-saturated designs and properties of the resulting designs are also proposed. Aggarwal and Gupta (2004) proposed construction method for multi-level super-saturated designs based on Galois field theory. Xu and Wu (2005) proposed construction methods for multi-level super-saturated designs inspired by Addelman- Kempthorne of orthogonal arrays and also proposed new lower bound for multi-level super-saturated designs.

Koukouvinov, Mantas and Mylona (2007) proposed mixed-level super-saturated designs by using supplementary difference sets with respect to the $E(f_{NOD})$ criterion. Nguyen and Cheng (2008) suggested the construction procedure for super-saturated designs from BIBD and also from regular graph designs when BIBD do not exist. Jones, Lin and Nachtsheim (2008) proposed a new class of super-saturated designs by using Bayesian D-optimality for arbitrary sample sizes and for any number of blocks of any size and also incorporate categorical factors with more than two levels.

Sun, Lin and Liu (2011) proposed equivalent conditions for two columns to be fully aliased and consequently proposed methods for constructing $E(f_{NOD})$ and chi-square optimal mixed level super-saturated designs without fully aliased columns via equidistant designs and difference matrices.

Gupta, Hisano and Morales (2011) proposed a systematic method of construction for optimal k-circulant multi-level supersaturated designs and also constructed two-level designs using resolvable BIBD. Liu, Y., and Liu, M.Q., (2011) proposed a new method for constructing mixed-level designs with relatively large number of levels avoiding the blind search and numerous calculations by computers. The goodness of the resulting supersaturated design is judged by the χ^2_{and} J_2 criteria's.

Mandal B.N., Gupta, V.K., and Prasas, R., (2011) proposed an algorithm to construct efficient balanced multi-level K-circulant supersaturated designs with 'm' factors and 'n' runs and they also constructed 60 factor and 10 levels multi-level supersaturated designs.

Liu and Liu (2012) generalized a method proposed by Liu and Lin to the mixed level case and also they proposed two new practical methods for constructing optimal mixed level supersaturated designs.

Gupta (2012) extended the work from two level to s-level balanced supersaturated designs. Gupta and Morales (2012) proposed tabu search method for constructing $E(s^2)$-optimal and minimax – optimal k-circulant supersaturated designs. Chatterjee, K., Koukouvinos C., Mantas, P., and Skountzou, A. (2012) proposed $E(f_{NOD})$-optimal multi-level supersaturated designs with a large number of columns based on the new supplementary difference sets method.

Hung, C., Lin, D.K.J., and Liu M.Q. (2012) proposed a new criterion for supersaturated designs with quantitative factors. Mbegbu, J.I., and Todo, C.O. (2012) proposed $E(s^2)$-optimal supersaturated designs with an experimental run size n=20 and number of factors m=57(multiple of 19). This construction is based on BIBD using a theorem proposed by Bulutoglu and Cheng.

Liu, Y., and Liu, M.Q. (2013) proposed complementary design method. The basic principle of this method is that for any existing $E(f_{NOD})$ optimal supersaturated whose $E(f_{NOD})$ value reaches its lower bound, its complementary design in the corresponding maximal balanced design is also $E(f_{NOD})$ optimal. This method applies to both symmetrical and asymmetrical cases.

Miller, A., and Tang, B. (2013) proposed supersaturated designs using minimal dependent sets (MDS) of columns in the design matrix. Ameen Saheb and Bhatracharyulu (2013, 2014) proposed two level supersaturated designs by using cyclic resolvable designs and also they proposed two new methods for constructing supersaturated designs using row-column and cyclic resolvable designs.

In this paper, an attempt is made propose to construct supersaturated designs using Balanced 'n' array Block Designs and Balanced Incomplete Block Designs.

3. Construction of Two Level Super-Saturated Designs

In this section, an attempt is made propose to construct two level supersaturated designs using Balanced n-array Block Design's (BnBD) and Balanced Incomplete Block Designs with suitable illustrations is presented. The $E(s^2)$-optimal values for the designs are also evaluated.

Method 3.1: Consider a complete set of (n-1) mutually orthogonal Latin squares (MOLS) of order n, where 'n' is prime or prime power. Arrange the (n-1) MOLS together to form an array of order n(n-1)x n consisting of elements $p_0, p_1, \ldots , p_{n-1}$ in each row. Replace 'n_1' of the p_i's by +1's, 'n_2' of the p_i's by -1's such that $n_1 + n_2 = n$. By considering each row of n(n-1) corresponds to a factor and each column as a design point the resulting design is a super-saturated design.

Note: When 'n' is even, complete set of mutually orthogonal Latin squares can be considered and when 'n' is odd either all (n-1) MOLS or a half of complete set, i.e. (n-1)/2 MOLS such that $^nC_2$ pairs of the 'n' elements occur exactly once in any two columned sub matrix of the array of order n(n-1)/2 x n can be considered.

Example 3.1: Consider the balanced 5 array design in 10 blocks replace p_0 and p_1 by '-1' and the other p_is '+1'. The resulting supersaturated design with 5 design points and 10 factors is given below.

$$
X = \begin{bmatrix}
-1 & -1 & +1 & +1 & +1 & -1 & -1 & +1 & +1 & +1 \\
-1 & +1 & +1 & +1 & -1 & +1 & +1 & -1 & -1 & +1 \\
+1 & +1 & -1 & -1 & +1 & +1 & -1 & -1 & +1 & +1 \\
+1 & -1 & -1 & +1 & +1 & +1 & +1 & +1 & -1 & -1 \\
+1 & +1 & +1 & -1 & -1 & -1 & +1 & +1 & +1 & -1
\end{bmatrix}
$$

The expected value of s^2 of the design is 3.6.

Method 3.2: Consider a Balanced incomplete Block Design with parameters v, b, r, k and λ whose incidence matrix is N^*. Obtain the matrix N_1 0's with -1 in N^* and obtain the matrix N_2 by replacing '1' with '-1' and 0's with 1's. A super saturated design X can be constructed with 'v' design points in '2b' factors as X = [N_1 N_2]′ where N_2 is replacing '1' by '-1' and '-1' by '1' of N_1.

Remark: When v=2k, X′X is singular then design matrix X has to be modified suitably given by X^*=[X J -J]′ where 'J' be a vector of unities.

Example 3.2: Consider a BIBD with parameters v=7=b, r=3=k, λ=1 whose incidence matrix is N^*. By delivering the matrices N_1 and N_2 by replacing the 0 by -1 and 0 by 1 and 1 by -1 respectively. The resulting supersaturated design is

$$X = \begin{bmatrix} +1 & +1 & -1 & -1 & +1 & -1 & -1 & -1 & -1 & +1 & +1 & -1 & +1 & +1 \\ +1 & -1 & +1 & -1 & -1 & +1 & -1 & -1 & +1 & -1 & +1 & +1 & -1 & +1 \\ -1 & +1 & -1 & +1 & -1 & +1 & -1 & +1 & -1 & +1 & -1 & +1 & -1 & +1 \\ -1 & -1 & +1 & +1 & +1 & -1 & -1 & +1 & +1 & -1 & -1 & -1 & +1 & +1 \\ +1 & -1 & -1 & +1 & -1 & -1 & +1 & -1 & +1 & +1 & -1 & +1 & +1 & -1 \\ -1 & +1 & +1 & -1 & -1 & -1 & +1 & +1 & -1 & -1 & +1 & +1 & +1 & -1 \\ -1 & -1 & -1 & -1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 & -1 & -1 & -1 \end{bmatrix}$$

The expected value of s^2 of the design is 4.65.

4. Concluding Remarks

Using the proposed methods, it is possible to estimate maximum number of main effects when compared to any other super-saturated designs. It found that the proposed methods of supersaturated designs are more efficient than the existed Booth and Cox designs.

References

Aggarwal, M.L., and Gupta, S. A new method of construction of multi-level supersaturated designs. *Journal of Statistical Planning and Inference.* 2004. 121; 127-134.

Ameen Saheb, Sk., and Bhatra Charyulu, N.Ch. Method for the construction of Super saturated design. *International Journal of Mathematical Archive.* 2013. 4 (12) 1-3.

Ameen Saheb, Sk., and Bhatra Charyulu, N.Ch. A New method for the construction of Super-Saturated design. *International Journal of Technology and Engineering Science.* 2013. 1 (6) 622-624.

Ameen Saheb, Sk., and Bhatra Charyulu, N.Ch. Construction of New Series of Super-Saturated Designs. *Research Journal of Mathematical and Statistical Sciences.* 2014. 2 (3) 7-8.

Booth, K.H.V., and Cox, D.R. Some systematic supersaturated designs. *Technometrics.* 1962. 4; 489-495.

Chatterjee, K., Koukouvinos, C., Mantas, P., and Skountzou, A. A general construction of E (fNOD)-optimal multi-level supersaturated designs. *Journal of Statistical Planning and Inference.* 2012. 142; 1092-1107.

Cheng, C.S. E(s2) Optimal supersaturated designs. *Statistica Sinica.* 1997. 7; 929-939.

Deng, L.Y., Lin, D.K.J., and Wang, J.N. A resolution rank criterion for supersaturated designs. *Statistica Sinica.* 1999. 9; 605-610.

Fang, K.T., Lin, D.K.J., and Ma, C.X. On the construction of multi-level supersaturated designs. *Journal of Statistical Planning and Inference.* 2000. 86; 239-252.

Gupta, S., and Chatterjee, K., Supersaturated designs: A review. *Jour. Comb. Info. Sys. Sci.* 1998. 23 (4) 475-488.

Gupta, S., and Morales, L.B. Constructing E(s2)-optimal and minimax-optimal k-circulant supersaturated designs via multi-objective tabu search. *Journal of Statistical Planning and Inference.* 2012. 142; 1415-1420.

Jones, B.A., Nachtsheim, C.J., and Lin, D.K.J. Bayesian D-optimal supersaturated designs. *Journal of Statistical Planning and Inference.* 2008. 138; 86-92.

Koukouvinos, C., Mantas, P., and Mylona, K. A general construction of E(fNOD) optimal mixed-level supersaturated designs. *Sankhya.* 2007. 69; 358-372.

Lin, D.K.J. A new class of Supersaturated designs. *Technometrics.* 1993. 35; 28-31.

Lin, D.K.J. Generating Systematic Supersaturated Designs. *Technometrics.* 1995. 37; 213-225.

Liu, Y., and Liu, M.Q. Construction of equidistant and weak equidistant supersaturated designs. *Metrika.* 2012. 75; 33-53.

Liu, Y., and Liu, M.Q. Construction of supersaturated design with large number of factors by the complementary design method. *Acta Mathematicae Applicatae Sinica.* 2013. English Series. 29; 253-262.

Liu, Y.F., and Dean, A. k-circulant supersaturated designs. *Technometrics.* 2004. 46; 32-43.

Lu, X., and Zhang, Y. A new method in the construction of two-level supersaturated designs. *Journal of Statistical Planning and Inference.* 2000. 86; 229-238.

Luis B. Morales. Constructing Cyclic PBIBD(2) Through an Optimization Approach: Thirty-Two New Cyclic Designs. *Inc. J Combin Designs.* 2004. 13; 377-387.

Miller, A., and Tang, B. Finding MDS-optimal supersaturated designs using computer searches. *Journal of Statistical Theory and Practice.* 2013. 7 (4) 703-712.

Nguyen, N.K. A algorithm approach to constructing supersaturated designs. *Technometrics.* 1996. 38; 69-73.

Nguyen, N.K., and Cheng, C.S. New E(s2)-optimal supersaturated designs constructed from incomplete block designs. *Technometrics.* 2008. 50 (1) 26-31.

Plackett, R.L., and Burmann, J.P. The design of optimum multi-factorial experiments. *Biometrica.* 1946. 33; 305-325.

Satterthwaite, F. Random Balance Experimentation. *Technometrics.* 1959. 1; 111-137.

Sun, F., Lin, D.K.J., and Liu, M.Q. On construction of optimal mixed-level supersaturated designs. *Annals of Statistics.* 2011. 39; 1310-1333.

Tang, B., and Wu, C.F.J. A method for constructing supersaturated designs and its E(s2)-optimality. *Canadian Journal of Statistics.* 1997. 25; 191-201.

Xu, H., and Wu, C.F.J. Construction of optimal multi-level supersaturated designs. *Annals of Statistics.* 2005. 33; 2811-2836.

Yamada, S., and Lin, D.K.J. Construction of mixed-level supersaturated designs. *Metrika.* 2002. 56; 205-214.