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Finitesimale Deformation in a Rotating Disc Having Variable Density Parameter

Pankaj Thakur

Department of Mathematics, Indus International University, Bathu, Una, Himachal Pradesh, India

Correspondence should be addressed to Pankaj Thakur, pankaj_thakur15@yahoo.co.in

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Abstract Finitesimale deformation in a rotating disc having variable density parameter has been studied by using Seth's transition theory. With the effect of density variation parameter, rotating disc requires lesser angular speed for compressible as well as incompressible materials. Circumferential stresses are maximum at the internal surface for incompressible materials as compared to compressible material. Rotating disc is likely to fracture by cleavage close to the bore.
Keywords *Disc, Stresses, Deformation, Yielding, Angular Speed, Density*

1. Introduction

Disc plays an important role in machine design. Stress analysis of rotating discs has an important role in engineering design. Rotating discs are the most critical part of rotors, turbines motor, compressors, high speed gears, flywheel, sink fits, turbo jet engines and computer's disc drive etc. Solutions for thin isotropic discs can be found in most of the standard elasticity and plasticity textbooks [1, 2, 3, 4, 5]. Chakrabarty [4] and Heyman [6] solved the problem for the plastic state by utilizing the solution in the elastic state and consider the plastic range with the help of Tresca's yield condition. Further, to obtain the elastic-plastic stresses, these authors matched the elastic and plastic stresses at the same radius $r = c$ of the disc. Perfectly elasticity and ideal plasticity are two extreme properties of the material and the use of ad-hoc rule like yield condition amounts to divide the two extreme properties by a sharp line, which is not physically possible. Seth's transition theory [7] does not required any assumptions like an yield criterion, incompressibility condition, associated flow rule and thus poses and solves a more general problem from which cases pertaining to the above assumptions can be worked out. This theory [7] utilizes the concept of generalized strain measure and asymptotic solution at critical points or turning points of the differential equations defining the deformed field and has been successfully applied to a large number of problems [7-29]. Seth [8] has defined the generalized principal strain measures as:

$$e_{ii} = \int_0^{\overset{A}{e}_{ii}} \left[1-2\overset{A}{e}_{ii}\right]^{\frac{n}{2}-1} d\overset{A}{e}_{ii} = \frac{1}{n}\left[1-\left(1-2\overset{A}{e}_{ii}\right)^{\frac{n}{2}}\right], \quad (i = 1, 2, 3) \tag{1}$$

Where n is the measure and $\overset{A}{e}_{ii}$ is the Almansi finite strain components. For n = -2, -1, 0, 1, 2 it gives Cauchy, Green Hencky, Swainger and Almansi measures respectively. In this research paper, we investigate the problem of finitesimale deformation in a rotating disc having variable density parameter by using Seth's transition theory. The density of disc is assumed to vary along the radius in the form:

$$\rho = \rho_0 \left(r / b\right)^{-m} \tag{2}$$

Where ρ_0 is the constant density at r = b and m *is* the density variation parameter. Results have been discussed and presented graphically.

2. Mathematical Model

We consider a thin annular disc of variable density with central bore of inner radius a and outer radius b is considered (Figure 1).

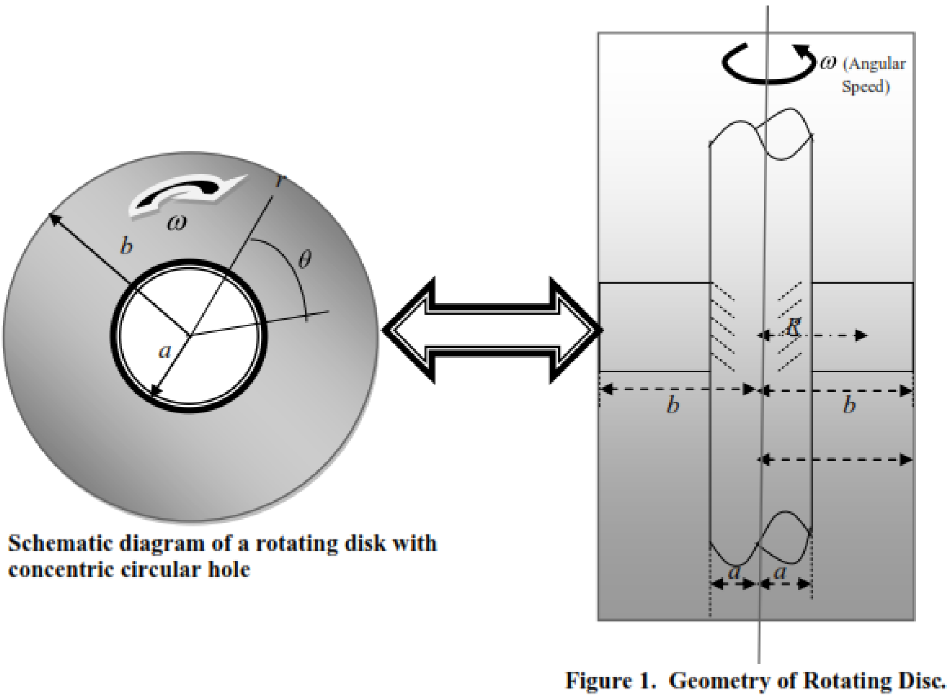

Figure 1: *Geometry of Rotating Disc.*

The disc is rotating with angular speed ω of gradually increasing magnitude about an axis perpendicular to its plane and passing through the center. The thickness of disc is assumed small so that the disc is effectively in a state of plane stress, that is, the axial stress T_{zz} is zero.

2.1 Formulation of the Problem

Displacement components in cylindrical polar co-ordinate (r, θ, z) are given by [8] as:

$u = r(1-\beta)$, $v = 0$, $w = dz$ (3)

Where β is position function, depending on $r = \sqrt{x^2+y^2}$ only and d is a constant.

The finite strain components are given by [8] as:

$$\overset{A}{e}_{rr} = \frac{1}{2}\left[1-(r\beta'+\beta)^2\right];\ \overset{A}{e}_{\theta\theta} = \frac{1}{2}\left[1-\beta^2\right];\ \overset{A}{e}_{zz} = \frac{1}{2}\left[1-(1-d)^2\right];\ \overset{A}{e}_{r\theta} = \overset{A}{e}_{\theta z} = \overset{A}{e}_{zr} = 0 \quad (4)$$

Where $\beta' = d\beta/dr$ and meaning of superscripts "*A*" is Almansi.

By substituting eq. (4) in eq. (1), the generalized components of strain become:

$$e_{rr} = \frac{1}{n}\left[1-(r\beta'+\beta)^n\right]; e_{\theta\theta} = \frac{1}{n}\left[1-\beta^n\right]; e_{zz} = \frac{1}{n}\left[1-(1-d)^n\right]; e_{r\theta} = e_{\theta z} = e_{zr} = 0 \quad (5)$$

The stress–strain relations for isotropic material are given [5]:

$$T_{ij} = \lambda\delta_{ij}I_1 + 2\mu e_{ij},\ (i, j = 1, 2, 3) \quad (6)$$

Where T_{ij} are stress components, λ and μ are Lame's constants, $I_1 = e_{kk}$ is the first strain invariant, δ_{ij} is the Kronecker's delta.

Equation (6) for this problem becomes

$$T_{rr} = \frac{2\lambda\mu}{\lambda+2\mu}\left[e_{rr}+e_{\theta\theta}\right]+2\mu e_{rr};\ T_{\theta\theta} = \frac{2\lambda\mu}{\lambda+2\mu}\left[e_{rr}+e_{\theta\theta}\right]+2\mu e_{\theta\theta};\ T_{r\theta} = T_{\theta z} = T_{zr} = T_{zz} = 0 \quad (7)$$

By substituting eq. (5) in eq. (7), the stresses are obtained as:

$$T_{rr} = \frac{2\mu}{n}\left[3-2C-\beta^n\left\{1-C+(2-C)(P+1)^n\right\}\right]$$

$$T_{\theta\theta} = \frac{2\mu}{n}\left[3-2C-\beta^n\left\{2-C+(1-C)(P+1)^n\right\}\right]$$

and $T_{r\theta} = T_{\theta z} = T_{zr} = T_{zz} = 0$ (8)

where C is the compressibility factor of the material in term of Lame's constant, given by $C = 2\mu/\lambda+2\mu$.

The equations of motion are all satisfied except:

$$\frac{d}{dr}(rT_{rr}) - T_{\theta\theta} + \rho\omega^2r^2 = 0 \quad (9)$$

Where ρ is the density of the material of the rotating disc.

By using eqs. (8) in eq. (9), we get a non- linear differential equation for β as:

$$(2-C)n\beta^{n+1}P(P+1)^{n-1}\frac{dP}{d\beta} = \left[\frac{n\rho\omega^2r^2}{2\mu}+\beta^n\left\{1-(P+1)^n-nP\left[1-C+(2-C)(P+1)^n\right]\right\}\right] \quad (10)$$

Where C is the compressibility factor of the material in term of Lame's constant, given by $C = 2\mu/\lambda + 2\mu$ and *P* is dependence function of β and β is dependence function of r only.
From eq. (10), the turning points of β are $P = -1$ and $\pm\infty$.

A. *Boundary Conditions* The boundary condition of the rotating disc is:

(i) $T_{rr} = 0$, $r = a$

(ii) $T_{rr} = 0$ $r = b$ (11)

Where T_{rr} denote stress along the radial direction.

B. *Solution of Problem* It has been shown that the asymptotic solution through the principal stress leads from elastic state to the plastic state (see Seth [7, 8], Gupta and Thakur [9-11] and Thakur Pankaj [12 - 29] at the transition point $P \to \pm\infty$. The transition function R is defined as:

$$R = \frac{nT_{\theta\theta}}{2\mu} = 3 - 2C - \beta^n\left[2 - C + (1-C)(P+1)^n\right] \tag{12}$$

Taking the logarithmic differentiating of eq. (12) with respect to *r*, we get:

$$\frac{d}{dr}(\log R) = \left(-\frac{n\beta^n P}{r}\right)\frac{\left[2 - C + (1-C)(P+1)^{n-1}\left\{(P+1) + \beta\frac{dP}{d\beta}\right\}\right]}{\left\{3 - 2C - \beta^n\left[2 - C + (1-C)(P+1)^n\right]\right\}} \tag{13}$$

By substituting the value of $dP/d\beta$ from eq. (10) into eq. (13) and by taking asymptotic value $P \to \pm\infty$, one gets after integration:

$$R = Ar^{\nu-1} \tag{14}$$

Where A is a constant of integration, which can be determined by boundary condition and by, $\nu = 1 - C/2 - C$ is the Poisson's ratio.
From eq. (12) and (14), it follows:

$$T_{\theta\theta} = \left(\frac{2\mu}{n}\right)Ar^{\nu-1} \tag{15}$$

By substituting eq. (15) into eq. (9) and using eq. (2), then integrating, we get:

$$T_{rr} = \frac{B}{r} + \left\{\frac{2\mu}{n\nu}\right\}Ar^{\nu-1} - \frac{\rho_0\omega^2 b^m r^{2-m}}{(3-m)} \tag{16}$$

Where B is a constant of integration, which can be determined by boundary condition.

By applying boundary condition from eq. (11) in eq. (16), we get: $A = \dfrac{\rho_0\omega^2 n\nu b^m\left(b^{3-m} - a^{3-m}\right)}{2\mu(3-m)\left(b^\nu - a^\nu\right)}$

and $B = \dfrac{\rho_0\omega^2 b^m a^{3-m}}{(3-m)} - \dfrac{\rho_0\omega^2 b^m\left(b^{3-m} - a^{3-m}\right)}{(3-m)\left(b^\nu - a^\nu\right)}a^\nu$.

By substituting the value of A and B into eqs. (15) and (16), we get:

$$T_{rr} = \frac{\rho_0 \omega^2 b^m}{(3-m) r}\left[\left(\frac{b^{3-m} - a^{3-m}}{b^\nu - a^\nu}\right)\left(r^\nu - a^\nu\right) - r^{3-m} + a^{3-m}\right] \tag{17}$$

$$T_{\theta\theta} = \frac{\rho_0 \omega^2 b^m \nu \left(b^{3-m} - a^{3-m}\right)}{(3-m)\left(b^\nu - a^\nu\right)} r^{\nu-1} \tag{18}$$

equations (17) and (18) gives elastic-plastic transitional stresses in a thin rotating disc of variable thickness with edge loading.

C. ***Initial Yielding of Rotating Disc*** It is seen from equ. (18) that $\left|T_{\theta\theta}\right|$ is maximum at the internal surface (*r* = *a*). Therefore yielding will take place at the inner surface and equ. (18) become:

$$\left|T_{\theta\theta}\right|_{r=a} = \left|\frac{\rho_0 \omega^2 b^m \nu \left(b^{3-m} - a^{3-m}\right)}{(3-m)\left(b^\nu - a^\nu\right)} a^{\nu-1}\right| \equiv Y(say)$$

and angular speed ω_i necessary for initial yielding is given by:

$$\Omega_i^2 = \frac{\rho_0 \omega_i^2 b^2}{Y} = \frac{(3-m)\left(b^\nu - a^\nu\right) b^2}{\nu\left(b^{3-m} - a^{3-m}\right) b^m a^{\nu-1}} \tag{19}$$

Where $\omega_i = \frac{1}{b}\Omega_i \left(Y/\rho_0\right)^{\frac{1}{2}}$. We introduce the following non-dimensional components as: *R* = *r*/*b*, $R_0 = a/b, \Omega^2 = \rho_0 \omega^2 b^2 / Y, \sigma_r = T_{rr}/Y, \sigma_\theta = T_{\theta\theta}/Y$. Eqs. (17), (18) and (19) become:

$$\sigma_r = \frac{\Omega_i^2}{(3-m) R}\left[\frac{\left(1 - R_0^{3-m}\right)}{\left(1 - R_0^\nu\right)}\left(R^\nu - R_0^\nu\right) - R^{3-m} + R_0^{3-m}\right] \tag{20}$$

$$\sigma_\theta = \frac{\Omega_i^2 \nu \left(1 - R_0^{3-m}\right)}{(3-m)\left(1 - R_0^\nu\right)} R^{\nu-1} \tag{21}$$

$$\Omega_i^2 = \frac{(3-m)\left(1 - R_0^\nu\right)}{\left(1 - R_0^{3-m}\right)\nu} R_0^{1-\nu} \tag{22}$$

D. ***Fully Plastic State of Rotating Disc*** The angular speed $\omega_f > \omega_i$ for which the rotating disc become fully plastic $\left(\nu \rightarrow 1/2 = 0.5\right)$ at the external surface *r* = b, equation (18) becomes

$$\left|T_{\theta\theta}\right|_{r=b} = \left|\frac{\rho_0 \omega^2 b^m \left(b^{3-m} - a^{3-m}\right)}{2(3-m)\sqrt{b}\left(\sqrt{b} - \sqrt{a}\right)}\right| \equiv Y^*(say)$$

and angular speed ω_f necessary for initial yielding is given by:

$$\Omega_f^2=\frac{\rho_0\omega_f^2b^2}{Y^*}=\frac{2(3-m)\sqrt{b}\left(\sqrt{b}-\sqrt{a}\right)b^2}{b^m\left(b^{3-m}-a^{3-m}\right)} \tag{23}$$

and stresses and angular speed give by eqs. (20), (21) and (23) for fully plastic state $(\nu\to 1/2=0.5)$ become:

$$\sigma_r=\frac{\Omega_f^2}{(3-m)R}\left[\frac{\left(1-R_0^{3-m}\right)}{\left(1-\sqrt{R_0}\right)}\left(\sqrt{R}-\sqrt{R_0}\right)-R^{3-m}+R_0^{3-m}\right] \tag{24}$$

$$\sigma_\theta=\frac{\Omega_f^2\left(1-R_0^{3-m}\right)}{2(3-m)\sqrt{R}\left(1-\sqrt{R_0}\right)} \tag{25}$$

$$\Omega_f^2=\frac{\rho_0\omega_f^2b^2}{Y^*}=\frac{2(3-m)\left(1-\sqrt{R_0}\right)}{\left(1-R_0^{\,3-m}\right)} \tag{26}$$

Neglects density parameter (m = 0), transitional stresses and angular speed from eqs. (20)- (22) becomes:

$$\sigma_r=\frac{\Omega_i^2}{3R}\left[\frac{\left(1-R_0^3\right)}{\left(1-R_0^\nu\right)}\left(R^\nu-R_0^\nu\right)-R^3+R_0^3\right] \tag{27}$$

$$\sigma_\theta=\frac{\Omega_i^2\nu\left(1-R_0^3\right)}{3\left(1-R_0^\nu\right)}R^{\nu-1} \tag{28}$$

$$\Omega_i^2=\frac{3\left(1-R_0^\nu\right)}{\left(1-R_0^3\right)\nu}R_0^{1-\nu} \tag{29}$$

and without density variation parameter, stresses and angular speed for fully plastic state from eqs. (24), (25) and (26) becomes:

$$\sigma_r=\frac{\Omega_f^2}{3R}\left[\frac{\left(1-R_0^3\right)}{\left(1-\sqrt{R_0}\right)}\left(\sqrt{R}-\sqrt{R_0}\right)-R^3+R_0^3\right] \tag{30}$$

$$\sigma_\theta=\frac{\Omega_f^2\left(1-R_0^3\right)}{6\sqrt{R}\left(1-\sqrt{R_0}\right)} \tag{31}$$

$$\Omega_f^2=\frac{\rho_0\omega_f^2b^2}{Y^*}=\frac{6\left(1-\sqrt{R_0}\right)}{\left(1-R_0^{\,3}\right)} \tag{32}$$

3. Numerically Discussion

For calculating the stresses based on the above analysis, the following values have been taken as C = 0.00, 0.25, 0.5, 0.75, m = 0, 1, 2 respectively. Curves have been drawn in figure 2 between angular speed Ω_i^2 required for initial yielding and various radii ratios $R_0=a/b$ for C = 0, 0.25, 0.5 at m = 0, 1, 2. It has been observed that the rotating disc made of incompressible material required higher angular speed for initial yielding as compared to disc made of compressible materials. With effect of density variation parameter, rotating disc requires lesser angular speed as compared to without density variation parameter. It can also be seen from Table 1 that for compressible material higher percentage increased in angular speed is required to become fully plastic as compared to rotating disc made of incompressible material.

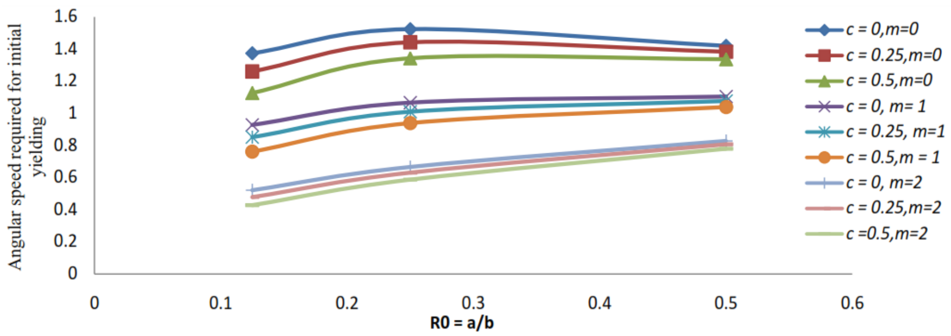

Figure 2: *Angular speed required for Initial Yielding State along the Radii Ratio Ro = a/b.*

Table 1: *Angular Speed Required for Initial Yielding and Fully Plastic State*

	Variable density parameter *m*	Compressibility of Material C	Angular Speed required for initial yielding (Ω_i^2)	Angular Speed required for fully-plastic state(Ω_f^2)	Percentage increase in Angular speed $\left(\sqrt{\frac{\Omega_f^2}{\Omega_i^2}}-1\right)\times 100$
$0.5 \le R \le 1.0$	0	0	1.420161	2.008411	18.92071307%
	1	0	1.104569	1.562097	18.92073204%
	2	0	0.828427	1.171573	18.92072679%
	0	0.25	1.383601	2.008411	20.48163765%
	1	0.25	1.076134	1.562097	20.48162673%
	2	0.25	0.8071	1.171573	20.48167691%
	0	0.5	1.336737	2.008411	22.57539772%
	1	0.5	1.039684	1.562097	22.57539993%
	2	0.5	0.779763	1.171573	22.57541301%

In figures 3 and 4, curve have been drawn between stresses and radii ratio $R = r/b$ for elastic-plastic transition state and fully plastic state. It has been seen circumferential stresses is maximum at the internal surface for incompressible materials ($C = 0$) as compared to compressible materials ($C = 0.25, 0.5$). Density variation parameter has a quit effect on circumferential stresses *i.e.* with the introduction of density variation parameter it decreases the values of circumferential stresses at the internal at the internal surface for transitional state and for fully plastic state. Rotating disc *is* likely to fracture by cleavage close to the bore.

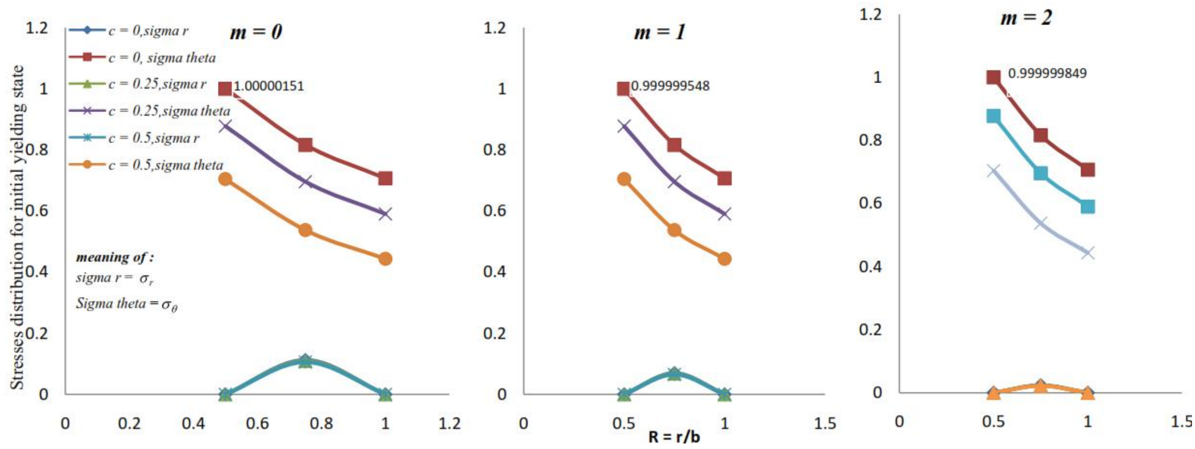

Figure 3: *Stresses Distribution in a Thin Rotating Disc for Initial Yielding State along the Radius Ratio R = r/b.*

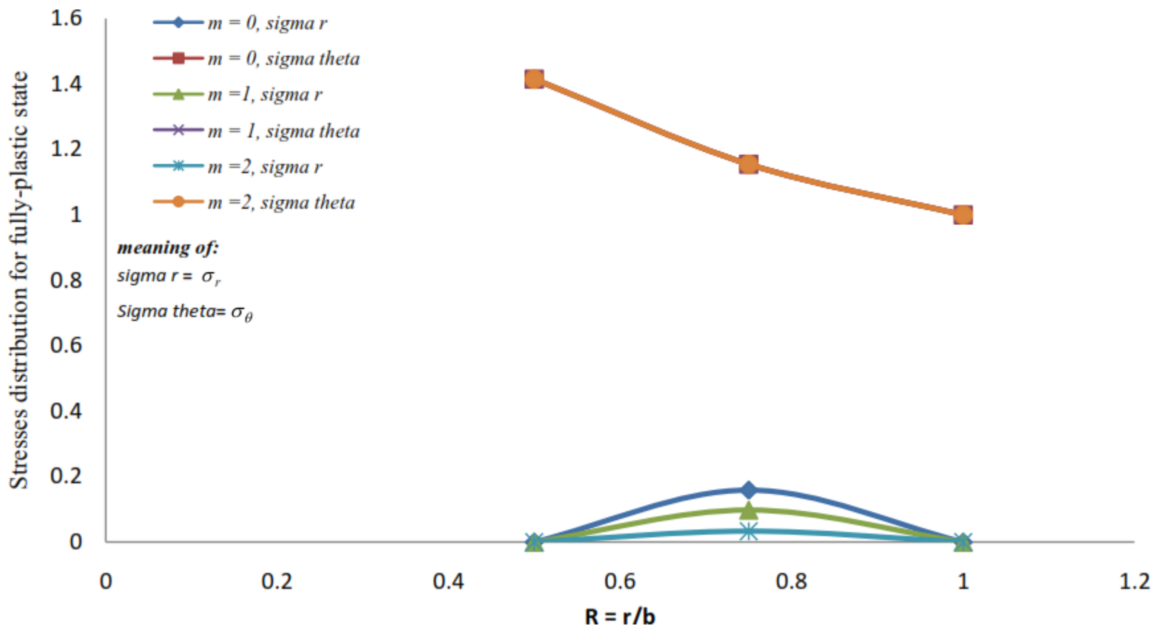

Figure 4: *Stresses Distribution in a Thin Rotating Disc for Fully-Plastic State along the Radius Ratio R = r/b.*

4. Conclusion

It has been observed that with the effect of density variation parameter, rotating disc requires lesser angular speed for compressible as well as incompressible material. Circumferential stresses are maximum at the internal surface for incompressible materials as compared to compressible material. Rotating disc is likely to fracture by cleavage close to the bore.

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